Cutting Convex Sets With Margin

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Background

- A geometric problem that arises in ***Density Estimation***
- Density Estimation = Distribution Learning w.r.t Total Variation
- Progress on problem $\Longrightarrow$ improved sample complexity bounds for optimal density estimators
- Based on joint works with

Olivier Bousquet, Mark Braverman, Klim Efremenko, Daniel Kane, and Gillat Kol



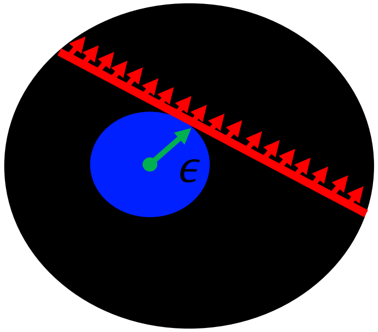


The Game

- Fix a norm $||\cdot||$ on $\mathbb{R}^d$ and $\epsilon > 0$
- $B(\vec{x}, r)$ = ball of radius r around $\vec{x}$

Cutting Game

- Player versus Adversary
- $B_0 = B(\vec{0}, 1)$
- At round $t = 0,1, \ldots$
  - Player picks $\vec{x}_t \in B_t$
  - Adversary picks halfspace H_t ***disjoint*** from $B(\vec{x}_t, \epsilon)$
  - Update $B_{t+1} = B_t \cap H_t$
- Player wants to reach $B_t = \emptyset$ as fast as possible



Example: ℓ_1 in 2 dimensions

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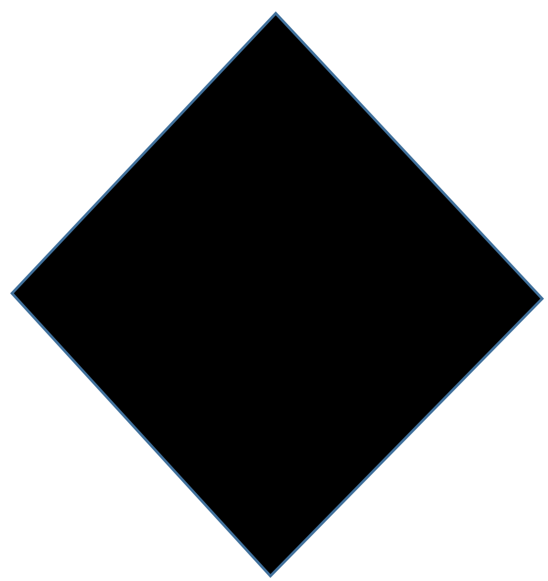

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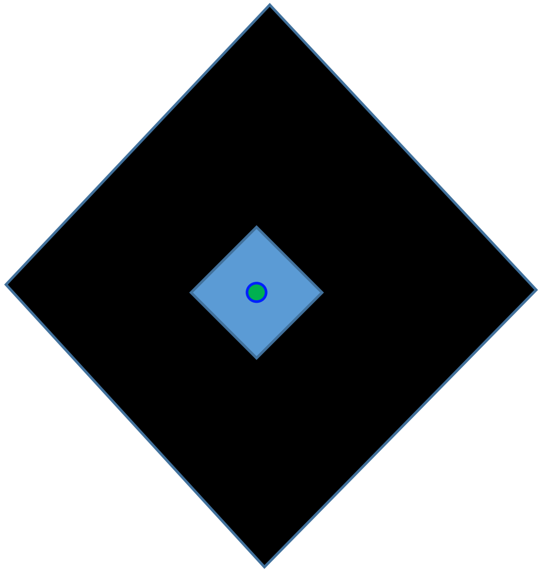
Round

1



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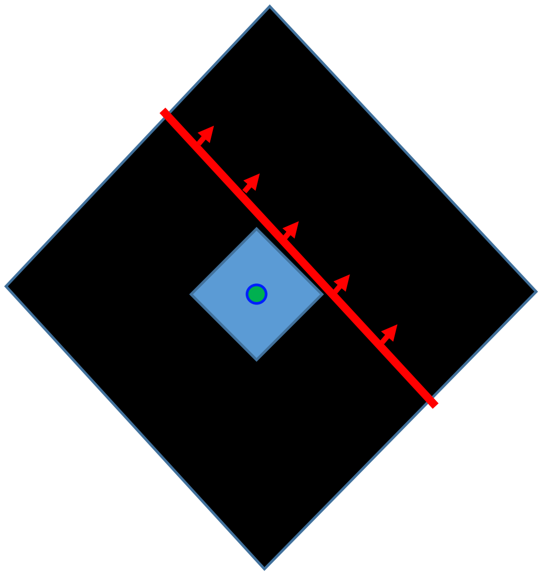

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Round

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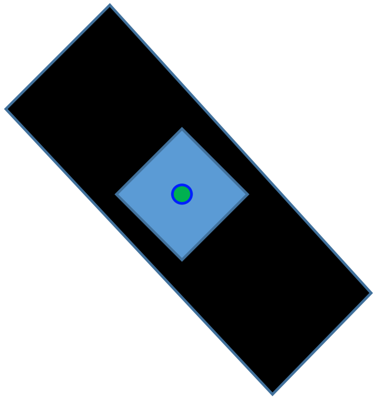

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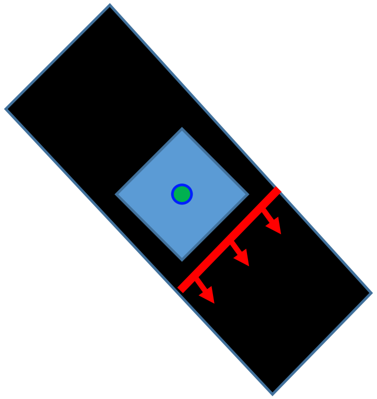
***Round* 2**



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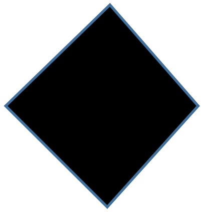

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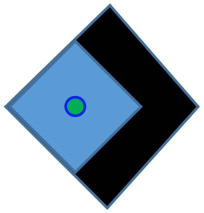

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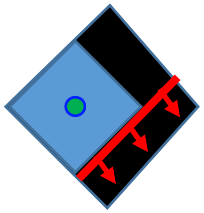

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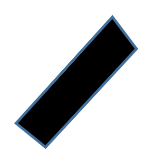

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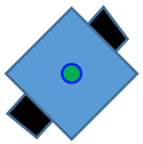

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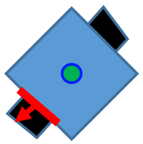

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Round

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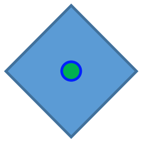

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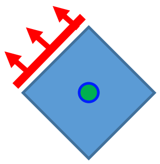

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Round

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Player wins at round $t = 6$

The Problem

$T(\|\cdot\|, d, \epsilon) :=$ min number of rounds T s.t. player has a strategy that guarantees $B_T = \emptyset$ against any adversary

Goal. Provide tight bounds on $T(\|\cdot\|, d, \epsilon)$

- Arbitrary norms? \\can be further extended to convex sets

- Norm $= \ell_p$?

Known Bounds

$T(\|\cdot\|, d, \epsilon) :=$ min number of rounds T s.t. player has a strategy that guarantees $B_T = \emptyset$ against any adversary

Goal. Provide tight bounds on $T(\|\cdot\|, d, \epsilon)$

- ($\forall$ norm $\|\cdot\|$): $T \leq O(d \log 1/\epsilon)$
- ($\exists$ norm $\|\cdot\|$): $T \geq \Omega(d \log 1/\epsilon)$ $\quad$ \\$\|\cdot\| = \ell_\infty$

Known Bounds

$T(\|\cdot\|, d, \epsilon) :=$ min number of rounds T s.t. player has a strategy
that guarantees $B_T = \emptyset$ against any adversary

Goal. Provide tight bounds on $T(\|\cdot\|, d, \epsilon)$

- ℓ_1: \\related to optimal density estimator
  - $T \leq O\left(\frac{\log d}{\epsilon^2}\right)$
  - $T \geq \Omega\left(\frac{\log d}{\epsilon}\right)$ \\ $d \geq \widetilde{\Omega}\left(\frac{1}{\epsilon^2}\right)$
- ℓ_p for $p \in (1,2]$: $T \leq O_p\left(\frac{1}{\epsilon^2}\right)$ \\indepenent of d
- ℓ_p for $p \in (2, \infty)$: $T \leq O\left(\frac{d^{1-\frac{2}{p}}}{\epsilon^2}\right)$

Thank you!